Relations between structural properties and synchronizability on local world dynamical networks*

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Abstract In this paper, the effects of various structural properties on the synchronization of coupled oscillators with local-world coupling configurations are investigated. It is found that for local world networks, the larger heterogeneity of the degree distribution, the enhanced interconnection of nodes, and the increased clustering do not improve the synchronizability of dynamical systems. On the contrary, the increase of the maximum betweenness centrality appears to be responsible for the decrease of the synchronizability.

Keywords: **complex dynamical network, local-world evolving model, synchronizability, structural properties, betweenness centrality.**

Many real-world complex systems, such as communication networks (Internet, World Wide Web), biological systems (protein interaction networks, neutral networks), traffic networks, social networks and so on, can be characterized by networks or graphs with complex topologies[1—5]. Thus, researchers from different disciplines have put their attention on this new research topic. The main goal of studying complex networks is to investigate how the networks' topological characteristics affect the dynamical functions taking place on such networks[1—4].

The dynamical systems of coupled oscillators have received a great deal of attention over the past two decades[6]. Especially, recent advances in complex network research have stimulated increasing interests in understanding the relationship between the topology and synchronization phenomena in dynamical networks[7—11]. The effects of various factors, such as node degree, heterogeneity, characteristic path length, clustering coefficient, and betweenness centrality on the synchronization, are investigated in dynamical systems with different coupling configurations.

In this paper, we consider a system of coupled oscillators on the local-world evolving networks, which is recently proposed in Ref. [12]. By varying the parameter—local world size M, the generated networks show a transition between exponential and scale-free networks with respect to the degree distribution $p(k)$. Here, $p(k)$ represents the probability that a randomly selected node in a network has k connections. The collective synchronization of dynamical systems with local-world coupling configurations is investigated based on a general framework for synchronization stability of coupled oscillator systems[13—15]. Here, the stability of the full synchronized state of coupled oscillators can be quantified through the eigenvalues of the coupling matrix, which represents the connection topology of the system. The research emphases are laid on how the synchronization is affected by various structural properties of the underlying network.

The rest of the paper is organized as follows: in Section 2, the local-world evolving network model and the general framework for synchronization stability of coupled oscillators are briefly discussed. In Section 3, the main part of the paper, investigation about the effects of the structural properties on the synchronizability is presented. Finally we conclude the whole paper in the last section.

1 Local-world evolving network model

The local-world evolving network model[12] is o-

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riginated form the BA scale-free model[5]. It inherits two ingredients: growth and preferential attachment, which are argued to be responsible for the emergence of the power-law degree distribution. However, difference exists in the preferential attachment mechanism: it involves local preferential attachment to capture the localization effect during the evolutions of real-world networks.

The iterative algorithm that can construct local-world evolving network is outlined as follows[12]:

(1) Growth: starting from a small number m_0 of isolated nodes, at each time step t, add a new node with $m(m \leqslant m_0)$ edges connecting to the network.

(2) Local preferential attachment: before connecting the new coming node to m existing nodes, randomly selecting $M(M \geqslant m)$ nodes referred to as the "local world"; then, adding edges between the new node and m nodes in this local world, the linking probability between any node i in the local world and the new node is

$$\Pi_{\text{local}}(i) = \frac{M}{m_0 + t} \frac{k_i}{\sum_{j \in \text{local}} k_j} \quad (1)$$

In BA model, preferential linking probability between the new node and an existing node i is

$$\Pi(i) = \frac{k_i}{\sum_j k_j} \quad (2)$$

Note that in Eq. (2) the summation in the denominator is valued over the whole network; but in local-world evolving model, the linking probability (Eq. (1)) is valued only within the local world of the new node. After T time steps, the algorithm results in a network with $N = m_0 + T$ nodes and $E = mT$ links.

Two special (limiting) cases exist in this model. When $M = m$, preferential attachment mechanism does not take effect, resulting in a network with the degree distribution following an exponential form: $p(k) \sim e^{-k/m}$. On the other hand, when $M \geqslant m_0 + T - 1$, which means M is always not smaller than the total number of nodes during network growth, the model reduces to the BA model with $p(k) \sim k^{-\gamma}$ and $\gamma = 3$. By varying the parameter M, networks evolved by this model represent a transition between these two extremes[12].

2 Synchronization of a system of coupled oscillators

In order to investigate the synchronization of coupled oscillators on the local world networks, an oscillator is placed at each node and a link between two nodes represents coupling between the two oscillators. Consider a network of N identical dynamical systems with symmetric coupling between oscillators[13—15], the equations of motion for the system are

$$\dot{x}_i = F(x_i) + c \sum_{j=1}^{N} L_{ij} H(x_j), \quad i = 1, 2, \cdots, N \quad (3)$$

where $\dot{x}_i = F(x_i)$ governs the dynamics of each individual node (i.e. with coupling strength $c = 0$) and $H(x)$ is the output function. The $N \times N$ matrix $\boldsymbol{L}$ is the coupling matrix, defined to be

$$L_{ij} = \begin{cases} k_i & \text{for } j = i \\ -1 & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Here, Λ_i denotes the neighbors of node i.

Because $\boldsymbol{L}$ is positive semidefinite and the sum of each raw of the matrix is zero, all its eigenvalues are $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$. If the network is connected, then $\lambda_2 > 0$. The ratio of the maximum eigenvalue λ_N to the smallest nonvanishing one λ_2 is used to determine the linear stability of the fully synchronized state $(x_1 = x_2 = \cdots = x_N)$ of dynamical elements. Since the ratio λ_N / λ_2 depends only on the topology of the underlying network, the impact of a particular coupling topology on the system's ability to synchronize can be characterized by this singular quantity: larger values of the ratio λ_N / λ_2 correspond to poor synchronizability, and *vice versa*[13—15].

3 Relations between structural properties and synchronizability

Having reduced the problem of the synchronizability of coupled oscillators to examine the eigenvalues of the corresponding coupling matrix, we now consider the synchronizability of dynamical system with local world coupling topologies.

The eigenvalue ratio λ_N / λ_2 for the local world networks with size $N = 2000$ and $m_0 = m = 2$ is obtained numerically and its behavior with the local world size M is exhibited in Fig. 1. The results are the average over 10 independent runs. As M is increased, the ratio λ_N/λ_2 is observed to increase, which implies the reduced synchronizability.

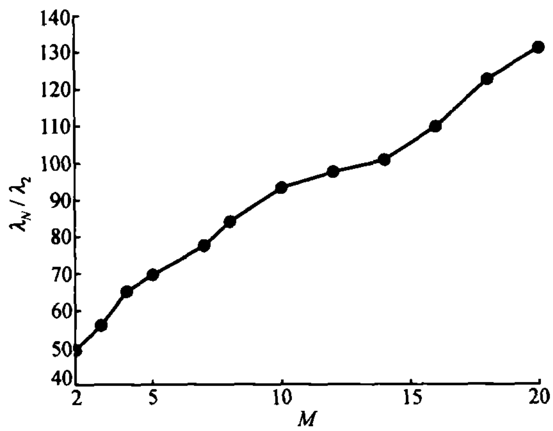

Fig. 1. Behaviors of the eigenvalue ratio λ_N/λ_2 with the parameter M in local world dynamical networks.

To study the relations between the structural properties and the synchronization on local world dynamical networks, we sequentially present numerical simulations to explore the changes on such structural properties with M, including the maximum node degree, heterogeneity, characteristic path length, global network efficiency, clustering coefficient and node betweenness centrality.

3.1 Synchronizability versus node degree

For node i, node degree k_i denotes the number of connections it owns. k_{max} is the maximum value among all the nodes and to some extent can demonstrate the inhomogeneity of the network's connectivity. Another measure, the variance

$$\sigma_k^2 = \frac{1}{N}\sum_{i=1}^{N}\left(k_i - \frac{1}{N}\sum_{i=1}^{N}k_i\right)^2 \tag{5}$$

of the degree distribution, is also used to describe the heterogeneity of a network.

Fig. 2 displays the dependencies of k_{max} and σ_k^2 on M for the local world networks with the same size $N = 2000$ and $m_0 = m = 2$ as those in Fig. 1. k_{max} and σ_k^2 grow as M is increased, which implies that the degree distribution becomes more heterogeneous and hub nodes with larger degrees appear. The results shown in Fig. 1 and Fig. 2 imply that the synchronizability on the local world network is suppressed as the heterogeneity of the degree distribution increases.

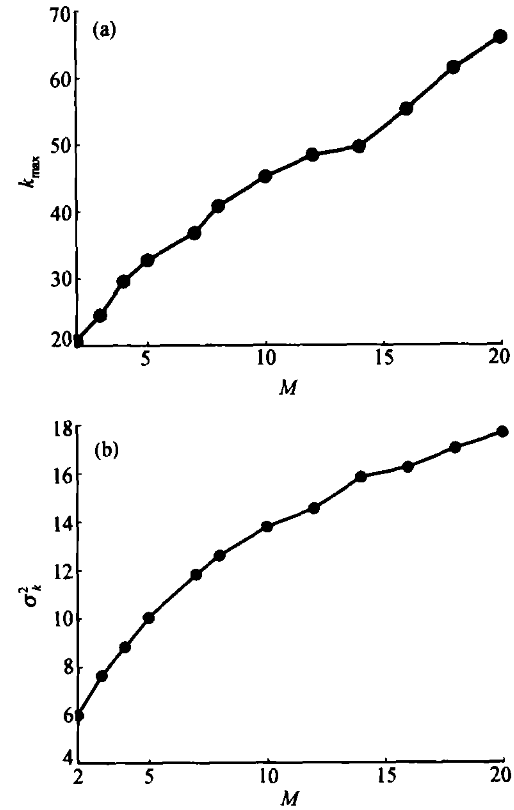

Fig. 2. Behaviors of (a) the maximum node degree k_{max} and (b) the variance σ_k^2 of the degree distribution with M in local world dynamical networks.

3.2 Synchronizability vs. characteristic path length

In Fig. 3, the behaviors of the characteristic path length D and global network efficiency E are shown as functions of the local world size M: as M is increased, D is observed to decrease and E to increase. These two measures characterize the ability of nodes to communicate with others and the efficiency of the information to be exchanged over the whole network[16]. They are defined as follows: d_{ij} denotes the length of the shortest path between node i and j, the efficiency ϵ_{ij} between i and j is assumed to be inversely proportional to the shortest distance, $\epsilon_{ij} = 1/d_{ij}$; accordingly,

$$D = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} d_{ij} \tag{6}$$

and

$$E = \frac{1}{N(N-1)}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{1}{d_{ij}} \tag{7}$$

The decrease of D and increase of E indicate that the network uses more efficiently the links and nodes, generating a more interconnected web. With a larger local world, the heterogeneity is increased, leading to smaller characteristic path length and improved communication efficiency. However, the synchronizability of the oscillators is reduced (Fig. 1 and Fig. 3).

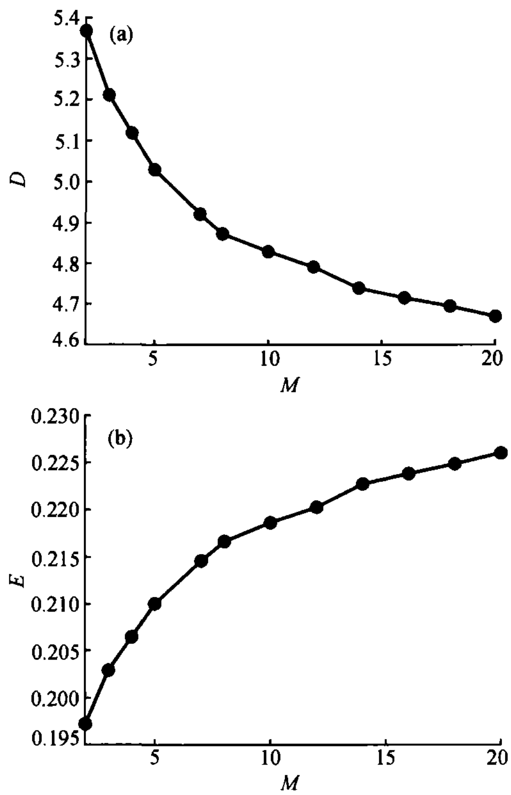

Fig. 3. Behaviors of (a) the characteristic path length D and (b) global network efficiency E with M in local world dynamical networks.

3.3 Synchronizability versus clustering

The relation between network synchronizability and clustering is also investigated. The average clustering coefficient C of a network quantifies the extent to which nodes adjacent to a given node are linked. For node i with degree k_i, E_i represents the set of its neighbor nodes and $|E_i|$ the number of edges between them, the local clustering coefficient C_i of node i is defined as

$$C_i = \frac{|E_i|}{k_i(k_i-1)/2} \tag{8}$$

and C is the average of C_i over all the nodes

$$C = \frac{1}{N}\sum_{i=1}^{N} C_i \tag{9}$$

As shown in Fig. 4, C increases with M, and thus higher clustering corresponds to poor synchronization on local world networks. The observation is consistent with the analytic results of the effect of clustering efficient on a highly-clustered scale-free network[17]: the more clustered the network, the poorer its synchronizability.

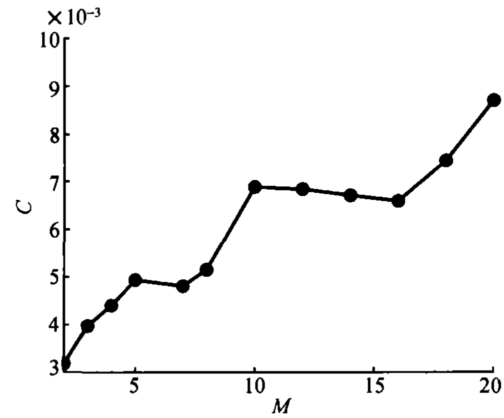

Fig. 4. Behaviors of the average clustering coefficient C with the parameter M in local world dynamical networks.

3.4 Synchronizability versus betweenness centrality

The betweenness centrality is regarded as a suitable predictor for the synchronizability on complex dynamical networks[7,8]. The betweenness centrality of node k is defined as:

$$B_k = \sum_{(i,j)} \frac{P_{ij}(k)}{P_{ij}} \tag{10}$$

where P_{ij} is the number of geodesic paths (shortest paths) between nodes i and j and $P_{ij}(k)$ is the number of these paths passing through node k. The summation is to be performed over all node pairs (i,j), such that $i \neq k$, $j \neq k$ and $i \neq j$, thus measuring the amount of communication traffic passing through a certain node[18—21]. B_{max} is the maximum value among all the N values of B_k and is shown to be highly related with the synchronizability of the system[8].

For local world dynamical networks, B_{max} at various values of the local world size M is obtained and displayed in Fig. 5. As the local world becomes larger, the maximum load on a node is increased. Fig. 5 together with Fig. 1 illustrates that the synchronizability is reduced as B_{max} is increased.

Why does the increase of the maximum node load B_{max} reduce the synchronizability (indicated by the increase of λ_N/λ_2)? The reason is rooted in the loss of information caused by overload on a few "center" nodes with B_{max}[7,8]. Since so many paths pass through the "center" nodes, they tend to get overloaded, consequently leading to the loss of the syn-

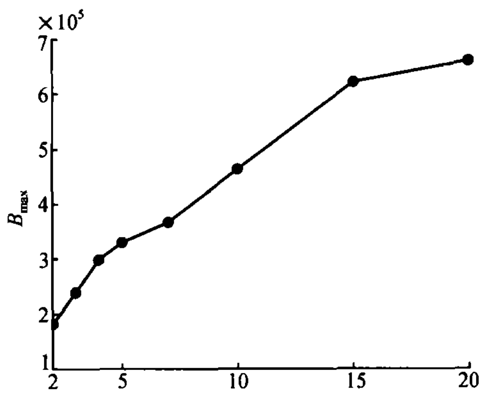

Fig. 5. Behaviors of the maximum node betweenness B_{max} with the parameter M in local world dynamical networks.

chronized state information to be exchanged between oscillators. Hence, the more traffic is centralized on a few nodes, the more difficult the network is to synchronize the oscillators.

3.5 Discussion

All the above figures show the relations between synchronization and structural properties on the local world networks: as the local world size M increases, the stability of synchronization is reduced with the following changes on structural properties: the maximum node degree, the heterogeneity of the degree distribution, the global communication efficiency, the average clustering coefficient and the maximum betweenness centrality are increased; and the characteristic path length is decreased.

For a dynamical system (3), the correlation between the structural properties of the coupling topology and its ability to synchronize the coupled oscillators can be illustrated by the rigorous bounds on the eigenvalue ratio λ_N/λ_2[7]

$$\left(1-\frac{1}{N}\right)\frac{k_{max}}{k_{min}} \leqslant \frac{\lambda_N}{\lambda_2} \leqslant (N-1)k_{max}B^e_{max}D_{max}D \tag{11}$$

where k_{max} and k_{min} are the maximum and minimum node degree, D_{max} is the maximum length of the shortest paths between any two nodes and B^e_{max} is the maximum of the normalized loads on links. Here, link load is defined as the number of the shortest paths between any two nodes that pass through the link. As far as the local world networks are concerned, $k_{min}=m$, and k_{max} increases with M, thus the ratio λ_N/λ_2 is increased. On the other hand, for the upper bound, more homogeneous network connectivity (smaller k_{max}), more balanced load distribution and less communication costs (smaller D_{max} and D) work together to make λ_N/λ_2 smaller and the oscillators are much easier to be synchronized.

Realizing that the synchronizability is strongly related with the load distribution of a network, some researchers have developed optimization strategies to improve networks' synchronizability by decreasing the concentrated load on a few heavy-loaded nodes[22,23]. Compared with the BA scale-free networks[5] (as a special case when $M\rightarrow\infty$), a smaller local world makes the load distribution more homogeneous and the corresponding network easier to achieve synchronization. This can be understood heuristically as follows: in the evolution of the network, due to the global preferential attachment, more links will be "absorbed" to a few "center" nodes, thus these "center" oscillators, interacting with a large number of other oscillators, tend to get overloaded by the traffic of communication passing through them. However, the existence of a local world with proper size will make links distributed between "small" nodes with a higher probability and consequently balance traffic load among the nodes. Thus, the local preferential linking mechanism (Eq. (1)) takes effects in a manner similar to the structural perturbation to a network, which aims at inhibiting overloaded traffic on some nodes[22,23].

4 Conclusion

In this paper, we have explored the relations between the structural properties and the synchronizability of coupled oscillators with local-world interaction topologies. We find that poor synchronizability is induced as the local world gets larger. In order to investigate how the structural properties of the underlying networks affect the synchronization of the system, we examined the changes of these properties as the local world size increases. The research results verified that larger heterogeneity of the degree distribution, shorter characteristic path length and higher clustering do not improve synchronizability. However, in the local world networks, the increase of the maximum betweenness centrality B_{max} appears to be responsible for the decrease of synchronizability.

Smaller characteristic path length D always implies more efficient communication between nodes. However, as the heterogeneity of the degree distribution increases, D decreases, but the traffic load may be centered on a few nodes, thus making the whole network reduce the ability to synchronize. These results strongly suggest that the balance between minimizing D to improve communication efficiency and decreasing concentrated load on a few nodes to enhance synchronizability should take into careful consideration in system design.

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